14/11/22

6. Gale duality & small polytopes

Q: In how far can combinatorial types polytopes be classified?

- recall 3D: $\sim \dfrac{1}{2^2 3^5 nm(n+m)} \binom{2m}{n+3}\binom{2n}{m+3}$

- but pretty hopeless in general dimensions
- let's try to classify $\underbrace{\text{small}}_{}$ polytopes
  ↳ few vertices relative to the dimension

- today: d ... dimension of P
  n ... number of vertices of P $(= f_0)$

- recall: – $n \geq d+1$
  – $n = d+1$ only for simplices

- what about $d+2$, $d+3$, ... ?
  – How many such polytopes exist (asymptotically)?
  – How can they be constructed / enumerated?

→ New technique: **Gale duality**
  – another way to "visualize" high-dimensional polytopes, but a bit more technical than e.g. Schlegel diagrams

- Gale duality is not specific to polytopes. Actually it is a duality for labelled point arrangements $p_1, \dots, p_n \in \mathbb{R}^d$.

d-dimensional n-point arrangement	$\Longleftrightarrow$	(n-d)-dimensional n-point arrangement
if n is not much larger than d	$\longrightarrow$	n-d is small (low-dimensional)

- There exist different forms of Gale duality adapted to different applications:

  linear, affine, spherical, ...

6.1 linear Gale duality

- fix a point arrangement $p_1, \dots, p_n \in \mathbb{R}^d$

  (e.g. vertices of a polytope)
- assume that p is full-dimensional

  (the p_i contain a basis of $\mathbb{R}^d$)

rank $X = d$ $\qquad\qquad\qquad$ $\dim U = d$

$$\begin{tikzcd}
\overset{(1)}{X :=} \begin{bmatrix} — p_1 — \\ — p_2 — \\ \vdots \\ — p_n — \end{bmatrix} \in \mathbb{R}^{n \times d} \arrow[r, "(2)"'] \arrow[d, leftrightarrow, "\text{Gale duality}"] & U := \operatorname{span} X \subseteq \mathbb{R}^n \arrow[d, leftrightarrow, "(3)"', "\substack{\text{duality:} \\ \text{orthogonal} \\ \text{complement}}"] \\
\underset{(5)}{X' :=} \begin{bmatrix} — q_1 — \\ — q_2 — \\ \vdots \\ — q_n — \end{bmatrix} \in \mathbb{R}^{n \times (n-d)} & U^\perp =: \operatorname{span} X' \arrow[l, "(4)"]
\end{tikzcd}$$

rank $X' = n - d$ $\qquad\qquad\qquad$ $\dim U^\perp = n - d$

Algorithm:

(1) write the p_i's as rows of a matrix X

(2) let U be the column span of X

(3) take the orthogonal complement of $U \longrightarrow U^\perp$

(4) find a matrix X' with column span $U^\perp$

(5) the rows of X' ar <u>a</u> linear Gale dual of p.

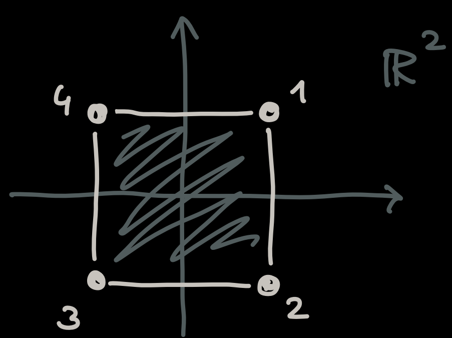
- Note: linear Gale dual is not unique since dependent on the choice of X'.

  Ex: Gale dual is unique up to linear transformations.

- if n is not much larger than d then the Gale dual is of a fairly small dimension.

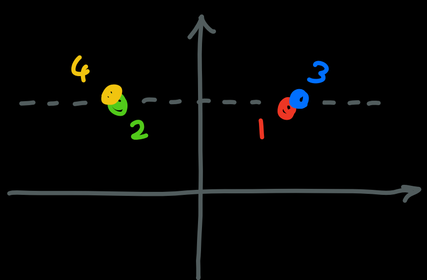
Example: square $[-1,1]^2$

$p_1 = (1,1)$, $p_2 = (1,-1)$, $p_3 = (-1,-1)$, $p_4 = (-1,1)$



$$X := \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} =: X'$$

in practice it is easier to go directly between the matrices



$q_1 = (1,1)$

$q_2 = (1,-1)$

$q_3 = (1,1)$

$q_4 = (1,-1)$

- the Gale dual itself is <u>not</u> a polytope
- points in the Gale dual can be on top of each other
- at least in this case: Gale dual seems to be contained in an affine subspace

6.2. affine Gale duality

- there is a way to "shave off one more dimension" of the Gale dual which comes also with other convenient properties.

Problem: • the linear Gale dual is not translation invariant
• but we mainly care about combinatorial types which are translation invariant

Solution: fix a canonical translation of point arrangement

e.g. $p_1 + \dots + p_n = 0$

$\longleftrightarrow$ $\vec{1} := (1, \dots, 1)^\perp$ is orthogonal to all columns of X

$\longleftrightarrow$ $\vec{1} \perp U$

$\longleftrightarrow$ $\vec{1} \in U^\perp$ (see square example)

- But if $\vec{1}$ is contained in $U^\perp$ for all point arrangement, then it carries no information and we can ignore it.
- Idea: take the orthogonal complement of U wrt. $\vec{1}^\perp$
- in practice: add a column $(1, \dots, 1)^\perp$ to X before converting to $U := \operatorname{span} X$.

$\longrightarrow$ $\operatorname{rank} X = d+1$

$\longrightarrow$ $\operatorname{rank} X' = n-d-1$

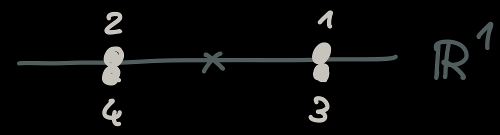
$\longrightarrow$ affine Gale dual is $(n-d-1)$-dimensional

Example: square again

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow X' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \longrightarrow$$

new column

Gale dual is now 1-dimensional



- the affine Gale dual is affinely full-dimensional

Ex: $q_1 + \cdots + q_n = 0$

- NOTE: when transforming back from q to p we do not add the extra column $(1, \dots, 1)$

  $\longrightarrow$ the duality becomes asymmetric

  $\longrightarrow$ there is an affine side (p) and a linear side (q)

Application: we can "classify" polytopes with $d+1$ vertices

- Gale dual is $n-d-1 = (d+1)-d-1 = 0$-dimensional
- 0-dimensional means $\mathbb{R}^0 := \{0\}$
- all points of the affine Gale dual are 0

  $\longrightarrow$ there is only one possible Gale dual

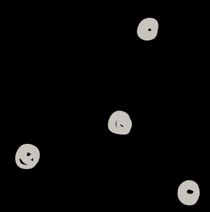
  $\longrightarrow$ there is only one possible such polytope for each $d \geq 1$ (up to affine transformation).

  $=$ d-simplex

6.3. Gale duals of polytopes

- Gale duals exist for all point arrangements

  → Q: can I tell whether it came from a polytope?



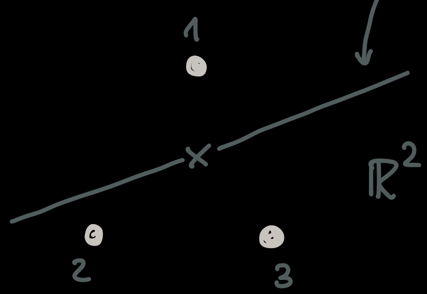
point arrangement whose points are not vertices of any polytope

Thm: $q_1, \dots, q_n \in \mathbb{R}^{n-d}$ is the (affine) Gale dual of a polytope (that is, its vertices) iff no point can be separated from the others by a central hyperplane. =: hyperplane that contains the origin

(proof later)

Example:

separates 1 from the others

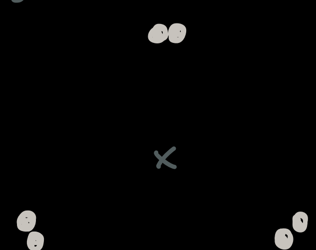


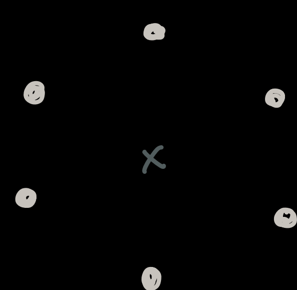
cannot be separated from the others because it is a double point.

(= Gale dual of square)

Example: other polytope Gale duals

triangle with double points





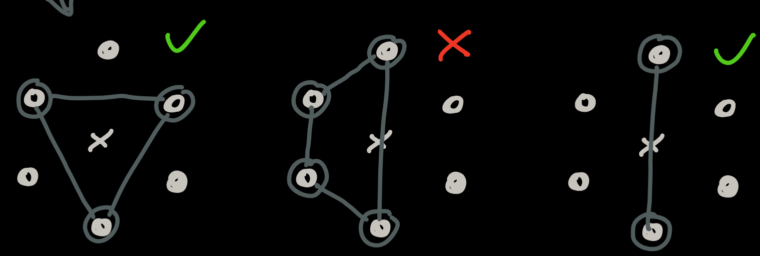
Ex: find the corresponding polytopes.

- We can actually read the full face-lattice from the Gale dual fairly easily.

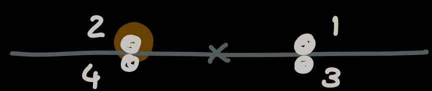
Thm: $S \subseteq \{1, \ldots, n\}$ corresponds to a face of P iff $\operatorname{conv}\{q_i \mid i \notin S\}$ contains the origin in its relative interior.

$:=$ interior relative to the affine hull.

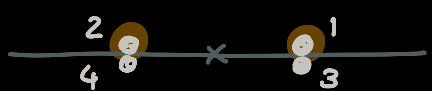
(proof later)



Example: square once again

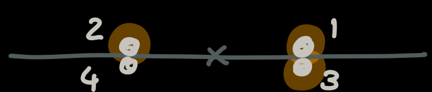


$\rightarrow$ $\operatorname{conv}\{p_1, p_3, p_4\}$ is no face



$\rightarrow$ $\operatorname{conv}\{p_3, p_4\}$ is a face

$\rightarrow$ $\operatorname{conv}\{p_2, p_4\}$ is no face



$\rightarrow$ $\operatorname{conv}\{p_4\}$ is a face

To prove the previous theorems it helps to clarify a conceptual point about what Gale duality "actually" does.

→ Gale duality (linear) swaps circuits with cocircuits

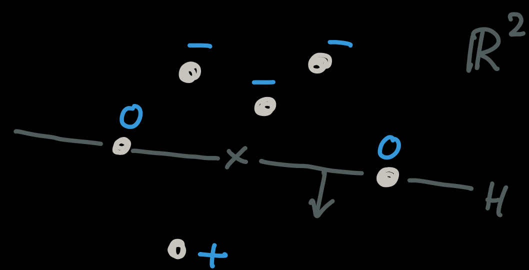
Def: a vector $v \in \{-,0,+\}^n$ is a $\hat{=}$ a sign assigned to each point of the arrangement

(i) circuit if there is a central hyperplane H that separates the $+$-points from the $-$-points and that contains all 0-points

(ii) cocircuit if there is a linear dependence

$$0 = \alpha_1 p_1 + \cdots + \alpha_n p_n$$

where α_i has sign v_i.



- the terminology "(co)circuit" comes from oriented matroid theory (an abstraction of linear algebra over ordered fields)

Thm: If $v \in \{-,0,+\}^n$ is a circuit for $p_1, \dots, p_n \in \mathbb{R}^d$ then v is a cocircuit for $q_1, \dots, q_n \in \mathbb{R}^d$ and vice versa.

Proof:

- suppose $\alpha_1 p_1 + \cdots + \alpha_n p_n = 0$ (cocircuit)

$$\Longleftrightarrow \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0 \qquad \alpha := (\alpha_1, \dots, \alpha_n)^T$$

$$\Leftrightarrow \quad X^T \alpha = 0$$
$$\Leftrightarrow \quad \alpha \perp \operatorname{span} X \quad \Leftrightarrow \quad \alpha \in U^\perp .$$

- suppose there is a central hyperplane H with normal vector $c \in \mathbb{R}^d$, then the entries of the corresponding circuit are the signs of $\beta_i := \langle c, p_i \rangle$.

$$\Leftrightarrow \quad \beta = Xc$$
$$\Leftrightarrow \quad \beta \in \operatorname{span} X \quad \Leftrightarrow \quad \beta \in U .$$

- since (linear) Gale duality swaps U and $U^\perp$ the previous equivalences show that it also swaps circuits and cocircuits. $\square$

<u>NOTE</u>: For <u>affine</u> Gale duality this still holds, but (co)circuits on the affine side (P) are defined slightly different:

(i) affine circuits are defined via general hyperplanes, <u>not</u> necessarily central.

(ii) affine cocircuits are defined using affine dependencies, <u>not</u> linear dependencies.

With this in place we can prove the previous results.

Thm: $q_1, \dots, q_n \in \mathbb{R}^{n-d}$ is the (affine) Gale dual of a polytope (that is, its vertices) iff no point can be separated from the others by a central hyperplane.

Proof:

- a vertex of a polytope cannot be written as the convex combination of other vertices.
- suppose $p_1 \in \operatorname{conv}\{p_2, \dots, p_n\}$

$$\Leftrightarrow \quad p_1 = \alpha_2 p_2 + \dots + \alpha_n p_n \qquad \alpha_i \geq 0, \ \sum \alpha_i = 1$$

$$\Leftrightarrow \quad 0 = -p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$$

$\longrightarrow$ $(-, 0/+, \dots, 0/+)$ is an (affine) cocircuit for p.

$\longrightarrow$ $(-, 0/+, \dots, 0/+)$ is a circuit for q.

- this means there is a central hyperplane that separates q_1 from the other points.

$\longrightarrow$ for a polytope this cannot happen. $\square$

Thm: $S \subseteq \{1, \dots, n\}$ corresponds to a face of P iff $\operatorname{conv}\{q_i \mid i \notin S\}$ contains the origin in its relative interior.

Proof:

- if $S \subseteq \{1, \dots, n\}$ corresponds to a face then there is a "touching" hyperplane, that is, it contains all points on the same side, and exactly p_i, $i \in S$ on it

$\longrightarrow$ there is an (affine) circuit $v \in \{-, 0, +\}^n$ of P with $v_i = 0$, $i \in S$, and $+$ everywhere else.

$\longrightarrow$ v is a cocircuit of the Gale dual of q:

$$0 = \alpha_1 q_1 + \cdots + \alpha_n q_n = \sum_{i \notin S} \alpha_i q_i \qquad \alpha_i > 0$$

- since $\alpha_i > 0$ we can normalize to $\sum \alpha_i = 1$
- so $0 \in \operatorname{conv} \{ q_i \mid i \notin S \}$
- since $\alpha_i > 0$ the origin is in the relative interior. $\square$

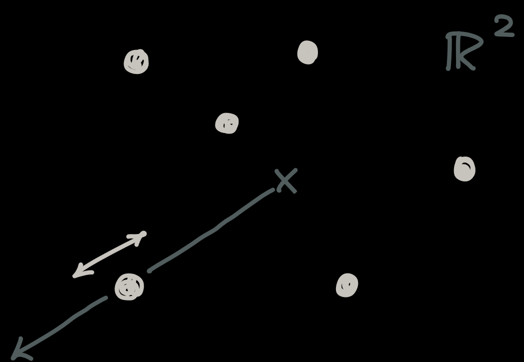
6.4. Polytopes with few vertices

- we are already dealt with $n = d+1$.
- <u>next</u>: $n = d+2$;
  - (affine) Gale dual is 1-dimensional
    $\longrightarrow$ q_i are on a line

  Problem: There are infinitely many ways to arrange points on a line.

  $\longrightarrow$ What differences are important for classification?

- We need to talk about one further type of Gale duality.

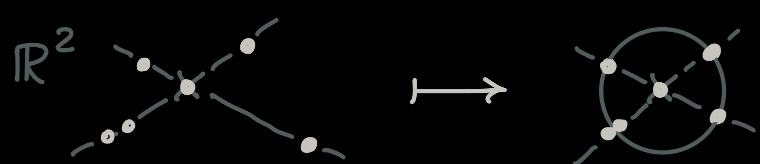


moving a point q_i along the ray λq_i, $\lambda > 0$ does not change whether it is the Gale dual of a polytope, nor its combinatorial type.

<u>Ex</u>: verify this using the previous theorems.

<u>Def</u>: spherical Gale diagram

:= projecting the non-zero points of the affine Gale dual onto the unit sphere.



→ the spherical Gale diagram contains all the information to reconstruct the combinatorial type (but not the polytope up to affine transformation)

- apply this to the case $n = d+2$:
  - unit sphere in $\mathbb{R}^1$: $\{+1, -1\}$

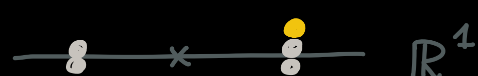
    → all points in 1-dimensional spherical diadram are $\in \{-1, 0, +1\}$.

<u>NOTE</u>:

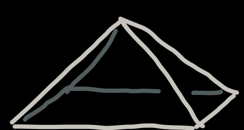
there need to be two points on each side of zero for this to be from a polytope.

→ $n \geq 4$

- <u>$d = 2$</u>: $n = 4$ there is a unique such polytope (the square)
- <u>$d = 3$</u>: $n = 5$ <u>two</u> polytopes

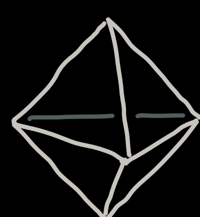


(*)



4-gonal pyramid

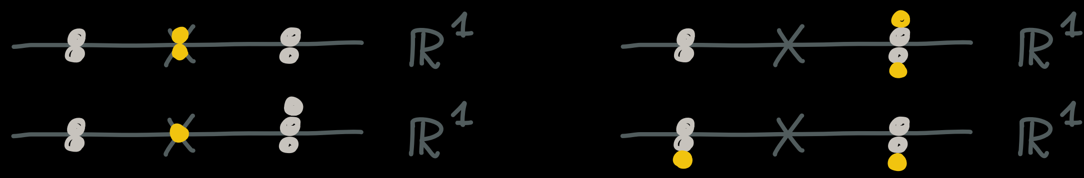
(*)



triangular bipyramid

<u>Ex</u>: verify that these are the polytopes to the diagrams.

- $\underline{d=4}$: $n=6$ $\underline{\text{four}}$ polytopes



these are : pyramids over $(*)$ and $(*)$

bipyramid over tetrahedron

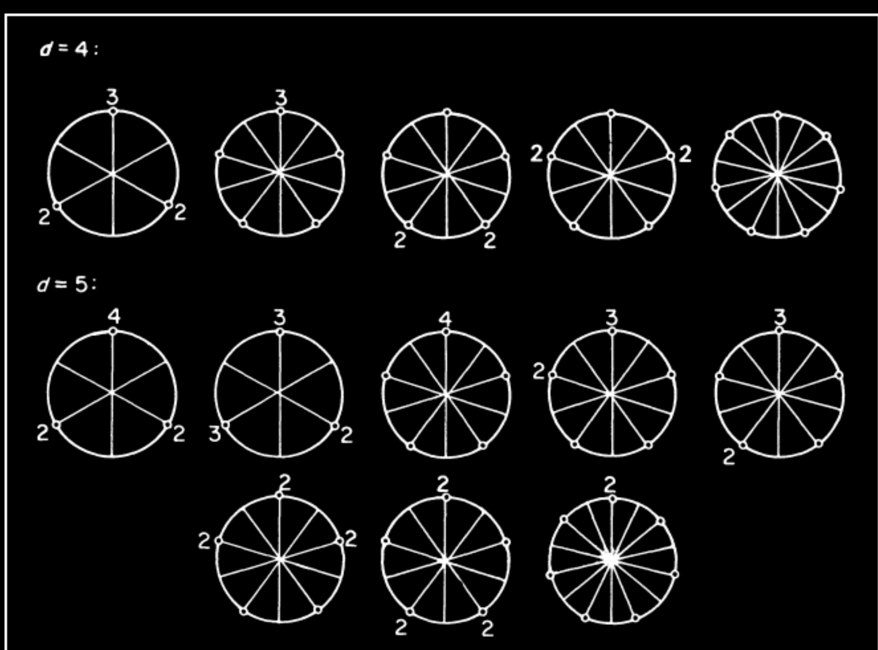
cyclic polytope $C_4(6)$

- $d=5$: $n=7$ $\underline{\text{six}}$ polytopes

<u>Ex</u>: find a closed formula for number of such polytopes in dimension d. $\in O(d^2)$

- from $n=d+3$ it starts to be real hard work.
- $\underline{n=d+3}$: exponentially many !!

$$\sim \frac{1}{d}\gamma^d \quad \text{with } \gamma \approx 2.8392\ldots$$



[Grünbaum]

spherical Gale diagrams of $d+3$ polytopes for $d=4$ and $d=5$ (Grünbaum)